Geometric Inflation and Late-Time Cosmic Acceleration from Embedded Spacetime Dynamics

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We develop a cosmological framework in which spacetime is treated as a four-dimensional manifold dynamically embedded in a higher-dimensional flat Minkowski background. Ultrarelativistic motion of the embedded manifold induces strong time-dilation effects between embedding time and proper time, generating a genuine phase of inflation with strict exponential expansion for comoving observers, without invoking an inflaton field or scalar potential. The inflationary phase satisfies the defining kinematic criteria, including a shrinking comoving Hubble radius, and admits a natural graceful exit as time dilation weakens. At late times, large-scale embedding dynamics give rise to a geometric expansion attractor that yields sustained cosmic acceleration without a bare cosmological constant. More generally, the attractor can be quasi-stationary, allowing a slow weakening of the effective acceleration rate while remaining non-phantom. Small deviations from uniform embedding motion excite long-wavelength co-dimensional modes that generate subdominant oscillatory corrections to the expansion rate. We derive the structure of linear perturbations arising from embedding fluctuations and show that they naturally produce nearly scale-invariant curvature perturbations with a suppressed tensor-to-scalar ratio. This framework provides a unified geometric origin for inflation, primordial structure, and late-time acceleration, without new fields or fine tuning.

I. INTRODUCTION

The accelerated expansion of the Universe appears twice in its history: an early inflationary epoch and the present phase of late-time cosmic acceleration. In the standard cosmological paradigm these phenomena are attributed to distinct sources—an inflaton field in the early Universe and dark energy or a cosmological constant at late times—whose microscopic origin remains unknown. Despite their phenomenological success, these explanations introduce new degrees of freedom and severe naturalness problems, including the origin of the inflaton potential, the smallness of the vacuum energy, and the apparent coincidence between matter and dark-energy densities.

An alternative viewpoint is that cosmic acceleration reflects the geometry of spacetime itself. In embedding-based approaches to gravity, spacetime is treated not as an abstract four-dimensional manifold but as a dynamical surface embedded in a higher-dimensional ambient space. In this picture, intrinsic curvature is inseparably linked to extrinsic geometry, and the dynamics of spacetime may depend on how it bends and moves within the higher-dimensional background. This idea was introduced by Regge and Teitelboim, who showed that Einstein gravity can be formulated as an embedding theory subject to additional constraints [1]. While embedding gravity has been explored primarily as a reformulation of general relativity or in brane-world scenarios [10, 12, 13], the direct cosmological role of collective bulk motion and co-dimensional fluctuations has remained largely untapped.

In this work we show that the embedding geometry of spacetime provides a unified explanation for both early- and late-time cosmic acceleration. Inflation arises from the ultrarelativistic motion of the spacetime manifold in the bulk, which induces strong gravitational time dilation between embedding time and proper time. This effect leads to strict exponential expansion as measured by comoving observers, without introducing an inflaton field or scalar potential.

At late times, large-scale embedding dynamics give rise to a geometric expansion attractor that produces sustained acceleration without invoking a bare cosmological constant. By a *geometric attractor* we mean a fixed point of the embedding-induced background evolution toward which the Hubble rate flows once matter and radiation have sufficiently diluted. In its simplest realization the attractor is stationary, corresponding to an approximately constant Hubble rate, but more generally it can be quasi-stationary, allowing slow temporal evolution of the effective acceleration while remaining non-phantom.

Small deviations from uniform embedding motion naturally excite long-wavelength co-dimensional modes, leading to subdominant oscillatory corrections to the expansion rate. A key feature of this framework is the natural separation between a dominant stationary component of the expansion, which sets the mean cosmic timescale, and a small oscillatory component associated with embedding fluctuations. For intuition, these may be referred to as the DC and AC components of the expansion, respectively. These components originate from different aspects of the embedding dynamics and are only weakly constrained by one another at the level of background evolution. As a result, the age of the Universe and the magnitude of late-time acceleration need not be tightly linked, in contrast to many dark-energy models.

This paper is organized as follows. In Sec. II we introduce the geometric framework of embedded spacetime and define the relevant kinematic quantities. In Sec. III we show how embedding-induced time dilation leads to a genuine phase of inflation with strict exponential expansion and a natural graceful exit. In Sec. IV we discuss the post-inflationary background evolution. In Sec. V we describe the emergence of late-time acceleration and the stationary plus oscillatory decomposition of the Hubble rate. In Sec. VI we analyze linear perturbations arising from embedding fluctuations and outline their observational consequences. We conclude in Sec. VII with a discussion of open questions and future directions.

A. Choice of time slicing: embedding time vs. brane proper time

An important subtlety in embedded cosmology is that the notion of "the size of the Universe at a given time" depends on the slicing used to foliate the spacetime worldvolume. Two natural choices exist. (i) *Constant embedding-time slices* $t =$ const correspond to the center-of-mass frame of the embedded manifold in the ambient space. (ii) *Constant proper-time slices* $\tau =$ const correspond to the physical rest frame of comoving observers on the spacetime manifold.

During the ultrarelativistic phase relevant for inflation, these two slicings are geometrically inequivalent. Embedding-time slices cut the worldvolume in sections whose physical size grows only linearly, $a(t) \propto t$, and therefore do not exhibit accelerated expansion. By contrast, proper-time slices intersect the same worldvolume at an angle that is nearly tangent to the ambient light cone. As a result, equal increments of proper time correspond to exponentially large changes in the spatial section, yielding strict exponential expansion $a(\tau) \propto e^{H\tau}$.

This geometric distinction resolves the apparent ambiguity in defining expansion during ultrarelativistic motion. Physical observables are always measured with respect to proper time τ, and only the τ-slicing corresponds to the expansion history seen by comoving observers. The embedding-time description serves as an auxiliary tool for understanding bulk kinematics and time-dilation effects, but does not define physical cosmological time.

This causal-structure picture of near-lightlike inflation and extrinsic-curvature-driven exit is illustrated in Fig. 1.

II. INFLATION FROM EMBEDDING-INDUCED TIME DILATION

In this section we show that a genuine phase of cosmic inflation arises naturally from the kinematics of embedded spacetime, without introducing an inflaton field, scalar potential, or slow-roll conditions. The mechanism relies on strong time dilation between embedding time and proper time, induced by ultrarelativistic motion of the spacetime manifold in the ambient space. Throughout this section we work in units where $c = 1$.

A. Ultrarelativistic bulk motion and lapse suppression

Consider an early configuration in which the spatial extent of the embedded spacetime manifold is microscopic, and its collective motion in one or more directions normal to the manifold is ultrarelativistic. Let $Y(t)$ denote a representative transverse embedding coordinate and define the corresponding bulk velocity

$$v \equiv \frac{dY}{dt}. \tag{1}$$

Along comoving worldlines on the manifold, the induced line element takes the form

$$d\tau^2 = dt^2\left(1 - v^2\right), \tag{2}$$

so that the lapse function relating proper time τ to embedding time t is

$$N(t) \equiv \frac{d\tau}{dt} = \sqrt{1 - v^2}. \tag{3}$$

When the bulk motion is ultrarelativistic, $v \simeq 1$, the lapse is strongly suppressed, $N \ll 1$. As a result, the proper time experienced by observers comoving with the spacetime manifold advances extremely slowly compared to the embedding time. Physically, this corresponds to strong gravitational time dilation between the ambient frame and the embedded spacetime. During this phase, neighboring regions of the manifold have very little proper time to interact causally, and the spacetime evolves almost freely as a coherent object in the higher-dimensional background.

B. Embedding-time expansion versus physical expansion

We define the expansion rate with respect to embedding time as

$$\mathcal{H}(t) \equiv \frac{1}{a}\frac{da}{dt}, \tag{4}$$

while the physical Hubble rate measured by comoving observers is defined with respect to proper time,

$$H(\tau) \equiv \frac{1}{a}\frac{da}{d\tau}. \tag{5}$$

Using the relation $d\tau = N(t)\,dt$, these quantities are related by the exact identity

$$H(\tau) = \frac{\mathcal{H}(t)}{N(t)}. \tag{6}$$

It is crucial to emphasize that accelerated expansion arises only when the Universe is foliated by constant-proper-time slices. On constant embedding-time slices, the scale factor grows only linearly, $a(t) \propto t$, and no inflation occurs. The physical expansion history measured by observers is therefore determined entirely by the behavior of $H(\tau)$, not by $\mathcal{H}(t)$ alone.

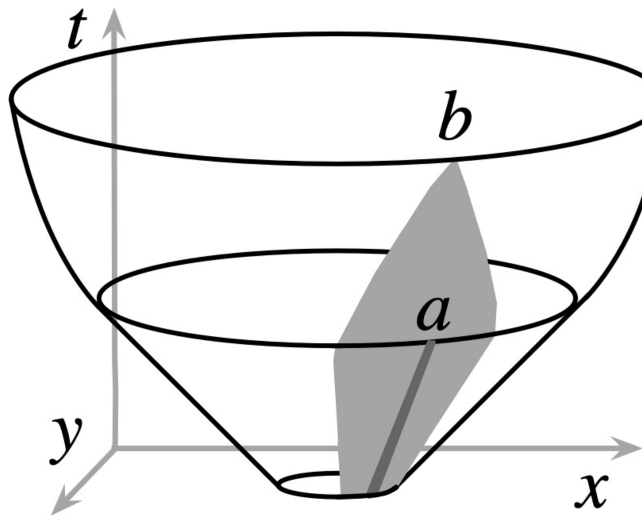

FIG. 1. Schematic illustration of inflation and graceful exit in embedded-spacetime cosmology. The spacetime manifold expands in a higher-dimensional flat ambient space and lies close to an ambient light cone during the inflationary phase. Strong time dilation ($N = d\tau/dt \ll 1$) magnifies modest embedding-time expansion into strict exponential growth in proper time. Inflation ends when extrinsic-curvature feedback slows bulk motion, restoring proper-time flow and causal communication. Points a (early) and b (late) illustrate compressed vs. expanded causal histories; the growth of causal regions with proper time is exponential due to the same geometric feedback.

C. Conditions for exponential inflation

A key observation is that the embedding-time expansion rate $\mathcal{H}(t)$ is controlled by smooth, large-scale embedding dynamics and need not be large or rapidly varying. In particular, during the ultrarelativistic phase the embedding equations admit solutions in which

$$\mathcal{H}(t) \simeq \mathcal{H}_{\rm inf} = \text{const}, \tag{7}$$

over an extended interval of embedding time. Such near-constant behavior is natural in the ultrarelativistic regime where extrinsic-curvature feedback is initially negligible, analogous to free coherent motion of the manifold in the bulk.

If, simultaneously, the lapse is strongly suppressed but slowly varying,

$$N(t) \simeq N_{\rm inf} \ll 1, \qquad \frac{\dot{N}}{N} \ll \mathcal{H}_{\rm inf}, \tag{8}$$

then the physical Hubble rate becomes

$$H(\tau) \simeq \frac{\mathcal{H}_{\rm inf}}{N_{\rm inf}} \equiv H_{\rm inf} = \text{const}. \tag{9}$$

Thus a modest, smooth expansion in embedding time is magnified into a large, approximately constant Hubble rate in proper time by the smallness of the lapse.

D. Exact exponential solution for the scale factor

We now show explicitly that a constant physical Hubble rate $H(\tau) = H_{\rm inf}$ implies strict exponential expansion. By definition,

$$H(\tau) = \frac{1}{a(\tau)} \frac{da(\tau)}{d\tau}. \tag{10}$$

If $H(\tau) = H_{\rm inf} = \text{const}$, this equation becomes

$$\frac{da(\tau)}{d\tau} = H_{\rm inf}\, a(\tau). \tag{11}$$

Dividing both sides by $a(\tau)$ and integrating yields

$$\int_{a_i}^{a(\tau)} \frac{da'}{a'} = \int_{\tau_i}^{\tau} H_{\rm inf}\, d\tau', \tag{12}$$

so that

$$\ln\left(\frac{a(\tau)}{a_i}\right) = H_{\rm inf}(\tau - \tau_i). \tag{13}$$

Exponentiating both sides gives the exact solution

$$a(\tau) = a_i \exp\left[H_{\rm inf}(\tau - \tau_i)\right]. \tag{14}$$

Choosing the normalization $a_i = 1$ at $\tau_i = 0$ yields

$$a(\tau) \propto e^{H_{\rm inf}\tau}. \tag{15}$$

The inflationary phase generated by embedding-induced time dilation is therefore *strictly exponential* in proper time, not merely approximately de Sitter. Because $H_{\rm inf}$ is approximately constant, the comoving Hubble radius decreases,

$$\frac{d}{d\tau}\left(\frac{1}{aH}\right) < 0, \tag{16}$$

satisfying the defining criterion for inflation and resolving the horizon and flatness problems in the usual way.

E. Number of e-folds and graceful exit

The total number of e-folds accumulated during inflation is

$$N_e \equiv \ln\left(\frac{a_f}{a_i}\right) = \int_{\tau_i}^{\tau_f} H(\tau)\, d\tau. \tag{17}$$

Using $d\tau = N(t)\, dt$ together with Eq. (6), this can be written purely in terms of embedding-time quantities,

$$N_e = \int \mathcal{H}(t)\, dt. \tag{18}$$

Thus sufficient inflation requires only that the embedding-time expansion rate remain approximately constant over a sufficiently long interval of embedding time. The duration of inflation in proper time can be arbitrarily short when the lapse is strongly suppressed.

Inflation ends naturally when the ultrarelativistic bulk motion slows and the lapse increases toward unity. As $N(t)$ grows, the physical Hubble rate $H = \mathcal{H}/N$ decreases even if $\mathcal{H}(t)$ remains smooth, and causal interactions between neighboring regions become efficient. The expansion then transitions smoothly to a decelerating Friedmann–Robertson–Walker phase without requiring a separate reheating field.

III. POST-INFLATIONARY BACKGROUND EVOLUTION

Following the ultrarelativistic phase described in Sec. III, the embedding dynamics naturally lead to a transition from inflation to a standard decelerating cosmological expansion. This transition occurs without the need for additional fields or fine-tuned conditions and is driven entirely by the relaxation of time-dilation effects.

A. Recovery of standard FRW evolution

Inflation ends when the collective bulk motion of the spacetime manifold slows and the lapse function $N(t)$ increases toward unity. As $N(t) \to 1$, the distinction between embedding time t and proper time τ becomes negligible. The physical Hubble rate reduces smoothly to

$$H(\tau) \simeq \mathcal{H}(t), \tag{19}$$

and the expansion history becomes governed primarily by the intrinsic energy content of the spacetime manifold. Provided that conventional matter and radiation are produced during or shortly after the end of the inflationary phase, the subsequent evolution follows the standard Friedmann–Robertson–Walker (FRW) behavior. In particular, the scale factor evolves as

$$a(\tau) \propto \begin{cases} \tau^{1/2}, & \text{radiation domination,} \\ \tau^{2/3}, & \text{matter domination,} \end{cases} \tag{20}$$

up to small corrections associated with residual embedding dynamics. Importantly, the inflationary mechanism described here does not modify the intrinsic gravitational field equations governing late-time FRW evolution. Because inflation is driven purely by time-dilation effects rather than by a large intrinsic energy density, standard cosmological behavior is recovered without tuning or additional assumptions.

B. Persistence of embedding memory

Although the lapse function relaxes to order unity after inflation, the embedding geometry of spacetime need not become dynamically trivial. The collective configuration of the spacetime manifold in the ambient space retains memory of its early evolution, encoded in conserved or slowly varying geometric quantities associated with the embedding. Small deviations from perfectly homogeneous embedding motion, seeded for example during the exit from inflation, can excite co-dimensional modes with extremely long wavelengths. Because these modes correspond to collective deformations of the spacetime manifold rather than to local field excitations, they are not efficiently damped by cosmic expansion and can persist over cosmological timescales.

During radiation and matter domination these residual embedding degrees of freedom are dynamically subdominant. However, they constitute a reservoir of geometric structure that can become relevant once conventional energy densities have sufficiently diluted.

C. Prelude to late-time acceleration

At the level of background evolution, the post-inflationary Universe therefore enters a regime characterized by a clear separation of timescales. Early inflation is controlled by the strong time dilation associated with ultrarelativistic bulk motion, while the intermediate cosmological eras are governed by standard FRW dynamics.

Late-time acceleration emerges when matter and radiation have diluted enough for the residual embedding geometry to dominate the expansion. In the next section we introduce an explicit geometric effective description of these late-time embedding dynamics and show how they naturally give rise to a stationary (or quasi-stationary) expansion attractor and a decomposition of the Hubble rate into dominant and subdominant components. This separation ensures that inflation, standard cosmological evolution, and late-time acceleration arise as distinct dynamical regimes of a single underlying structure: a four-dimensional spacetime manifold evolving in a higher-dimensional flat background.

IV. GEOMETRIC ACTION AND LATE-TIME ACCELERATION

A. Origin of the embedding equations of motion

The cosmological dynamics discussed in this work arise from treating spacetime as a four-dimensional manifold dynamically embedded in a higher-dimensional flat background. The fundamental variables are the embedding functions $X^A(x^\mu)$, from which both the induced metric $g_{\mu\nu}$ and the extrinsic curvature $K^I_{\mu\nu}$ are constructed.

At the effective level, the dynamics may be obtained from a geometric action of the generic form

$$S[X^A] = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R + \mathcal{L}_{\rm geo}(K^I_{\mu\nu}) + \mathcal{L}_m\right], \quad (21)$$

where $\mathcal{L}_{\rm geo}$ denotes the leading large-scale invariants built from the extrinsic curvature. Actions of this type arise naturally in the theory of embedded and rigid manifolds and include, as limiting cases, Regge–Teitelboim embedding gravity and Polyakov-type rigidity terms.

Variation with respect to the embedding functions yields equations of motion of the schematic form

$$\nabla_\mu\left(\Pi_I^{\mu\nu}\partial_\nu X^I\right) = 0, \quad (22)$$

where $\Pi_I^{\mu\nu}$ is a geometric stress tensor constructed from $g_{\mu\nu}$ and $K^I_{\mu\nu}$. For homogeneous and isotropic embeddings, these equations reduce to coupled evolution equations for the scale factor $a(\tau)$ and the collective bulk coordinates describing motion in the normal directions.

Two structural features of these equations are essential for the cosmological behavior emphasized here. First, ultrarelativistic solutions generically exist in which the embedding-time expansion rate remains smooth while the lapse $N = d\tau/dt$ is strongly suppressed, leading kinematically to strict exponential expansion in proper time. Second, the geometric sector admits stationary or quasi-stationary solutions at late times, corresponding to fixed points of the embedding dynamics once matter and radiation dilute.

The inflationary phase therefore arises primarily from embedding-induced time dilation and does not rely on the detailed form of $\mathcal{L}_{\rm geo}$. By contrast, the existence of a sustained late-time accelerating phase requires additional large-scale geometric structure, which we capture minimally through extrinsic-curvature invariants or, equivalently, conserved geometric charges associated with the embedding.

B. Effective geometric action

We consider an effective action of the form

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R + \alpha\, K^I_{\mu\nu}K^{I\,\mu\nu} + \beta\, K^I K^I + \mathcal{L}_{\rm m}\right], \quad (23)$$

where R is the Ricci scalar of the induced metric $g_{\mu\nu}$, $K^I_{\mu\nu}$ is the extrinsic curvature in the Ith normal direction, $K^I \equiv g^{\mu\nu}K^I_{\mu\nu}$, and $\mathcal{L}_{\rm m}$ denotes the Lagrangian for conventional matter and radiation. The coefficients α and β parametrize the rigidity of the embedded spacetime manifold and have dimensions of length squared. Actions of this type are well known in the theory of embedded surfaces and rigid branes and arise naturally as effective descriptions of collective geometric degrees of freedom.

Throughout this work the ambient spacetime is taken to be flat; all curvature effects originate from the embedding of spacetime itself. The Einstein–Hilbert term governs intrinsic geometry, while the extrinsic curvature terms encode the resistance of the spacetime manifold to bending and deformation in the higher-dimensional background. We emphasize that the specific values of α and β are not essential for the qualitative results that follow. What matters is the structural fact that the geometric sector contributes terms proportional to H^2 to the background evolution, thereby admitting stationary or quasi-stationary solutions.

C. Homogeneous background equations

For homogeneous and isotropic embeddings, the extrinsic curvature in each normal direction is proportional to the induced metric,

$$K^I_{\mu\nu} \propto H\, g_{\mu\nu}, \quad (24)$$

so that the quadratic extrinsic curvature invariants contribute effective terms scaling as H^2. Varying the action (23) with respect to the induced metric yields modified Friedmann equations of the schematic form

$$3M_{\rm Pl}^2 H^2 = \rho_{\rm m} + \rho_{\rm r} + \rho_{\rm geo}, \quad (25)$$

where $\rho_{\rm m}$ and $\rho_{\rm r}$ denote the usual matter and radiation energy densities, and $\rho_{\rm geo}$ is an effective geometric contribution sourced by the extrinsic curvature sector.

A key feature of the geometric contribution is the appearance of an integration constant or attractor solution corresponding to a stationary expansion rate. Once matter and radiation have sufficiently diluted, the background evolution approaches

$$H(\tau) \to H_{\rm stn} = \text{const}. \quad (26)$$

More generally, slow evolution of background embedding parameters yields a quasi-stationary weakening attractor, for example

$$H(\tau) \simeq H_{\rm stn}\left(1 - \epsilon\ln(\tau/\tau_0)\right), \quad |\epsilon| \ll 1. \quad (27)$$

This remains strictly accelerating ($\ddot{a} > 0$) and non-phantom. This solution represents a de Sitter–like phase driven entirely by embedding geometry, without invoking a bare cosmological constant or vacuum energy. Physically, it corresponds to a state in which the large-scale extrinsic geometry of spacetime supports a persistent expansion rate even in the absence of significant matter or radiation.

D. Stationary and oscillatory components of the expansion

While the stationary solution (26) controls the dominant late-time behavior, small departures from perfectly homogeneous embedding motion can excite codimensional modes with extremely long wavelengths. These modes introduce subdominant time-dependent corrections to the expansion rate. It is therefore natural to decompose the late-time Hubble parameter as

$$H(\tau) = H_{\rm stn} + H_{\rm osc}(\tau), \tag{28}$$

where $H_{\rm stn}$ is the stationary (or slowly varying) geometric component and $H_{\rm osc}$ encodes oscillatory or slowly varying modulations.

For intuition, these may be referred to as the DC and AC components of the expansion, respectively. A representative phenomenological template for the oscillatory contribution (expressed in redshift) is

$$H(z) = H_{\rm stn}\left[1 + A\cos(\omega \ln(1+z) + \phi)\right], \tag{29}$$

with amplitude $A \lesssim 0.01$, frequency $\omega \sim 1$–3, and phase ϕ determined by the properties of the excited embedding modes. Such small coherent modulations are testable with upcoming BAO and supernova surveys (e.g., DESI, Euclid).

The smallness of the oscillatory contribution is parametrically small,

$$|H_{\rm osc}| \ll H_{\rm stn}, \tag{30}$$

and therefore modulates rather than drives the accelerated expansion.

The DC component controls the overall acceleration and sets the cosmic timescale, while the AC component reflects residual geometric fluctuations seeded during the exit from inflation. Crucially, the stationary and oscillatory components arise from different aspects of the embedding geometry. The stationary component $H_{\rm stn}$ reflects a global geometric attractor or conserved quantity associated with large-scale embedding dynamics, while the oscillatory component $H_{\rm osc}$ is seeded by small nonuniformities, for example during the exit from inflation. As a result, the characteristic timescale of cosmic acceleration and the amplitude of oscillatory corrections need not be tightly correlated.

E. Interpretation and robustness

In the embedded-spacetime framework, sustained late-time acceleration is controlled primarily by the stationary component $H_{\rm stn}$. The oscillatory component represents a modulation of the expansion rather than its source and cannot by itself account for the observed magnitude of cosmic acceleration. This separation clarifies the physical role of embedding fluctuations and avoids confusion with models in which oscillations are invoked as the primary driver of acceleration.

The action (23) should be regarded as an effective description. Whether the stationary late-time attractor arises from explicit extrinsic curvature invariants, as written here, or from constraint dynamics and conserved geometric quantities in the spirit of Regge–Teitelboim embedding gravity remains an open question. In either case, the emergence of a stationary or quasi-stationary expansion phase appears to be a generic feature of embedded spacetime dynamics.

F. Weakening acceleration versus future contraction

Recent phenomenological analyses have emphasized the possibility that the effective late-time cosmic acceleration may be weakening, rather than remaining strictly constant. It is therefore essential to distinguish carefully between three logically distinct dynamical regimes: (i) sustained accelerated expansion, (ii) weakening acceleration, and (iii) genuine deceleration or contraction.

Although these possibilities are often conflated in observational discussions, they correspond to sharply different conditions in the present geometric framework.

a. Acceleration, deceleration, and contraction. The background expansion is governed by the scale factor $a(\tau)$ and the Hubble parameter

$$H(\tau) \equiv \frac{\dot a}{a}, \tag{31}$$

where overdots denote derivatives with respect to proper time τ. The condition for accelerated expansion is

$$\ddot a > 0 \quad \Longleftrightarrow \quad \dot H + H^2 > 0. \tag{32}$$

Deceleration corresponds to $\ddot a < 0$, while true contraction requires the stronger condition

$$H(\tau) < 0. \tag{33}$$

A weakening of acceleration therefore does *not* imply contraction unless $H(\tau)$ itself crosses zero.

b. Stationary and weakening geometric attractors. In the simplest realization of the geometric sector discussed in Sec. V, the late-time evolution approaches a stationary attractor,

$$H(\tau) \to H_{\rm stn} = {\rm const} > 0, \tag{34}$$

which yields eternal accelerated expansion with

$$\frac{\ddot a}{a} = H_{\rm stn}^2. \tag{35}$$

More generally, however, the attractor may be *quasi-stationary*, allowing a slow temporal drift,

$$H(\tau) = H_{\rm stn}(\tau), \qquad \left|\frac{\dot H_{\rm stn}}{H_{\rm stn}^2}\right| \ll 1. \tag{36}$$

In this regime the expansion remains accelerated provided

$$\dot H_{\rm stn} > -H_{\rm stn}^2, \tag{37}$$

even though the magnitude of $H(\tau)$ decreases slowly with time. This corresponds to a *weakening-attractor* phase: cosmic acceleration persists, but its effective strength gradually diminishes. Importantly, such behavior is generic in geometric systems with large-scale collective degrees of freedom. Slow drift of the attractor reflects gradual redistribution of geometric momentum or rigidity in the embedding sector rather than the onset of instability or fine tuning.

c. Role of oscillatory embedding modes. The oscillatory component introduced in Sec. V,

$$H_{\rm osc}(\tau) = \varepsilon\cos(\omega\tau + \phi), \qquad \varepsilon \ll H_{\rm stn}, \tag{38}$$

modulates the expansion rate but does not drive a transition to deceleration or contraction. To leading order,

$$\dot H_{\rm osc} \sim \varepsilon\omega, \tag{39}$$

which remains subdominant so long as

$$\varepsilon\omega \ll H_{\rm stn}^2. \tag{40}$$

Thus, long-wavelength embedding modes generically produce small ripples in the expansion history rather than qualitative changes in its long-term behavior.

d. Conditions for future contraction. A future contracting phase requires a much stronger condition: the stationary component itself must cross zero,

$$H_{\rm stn}(\tau_*) = 0, \tag{41}$$

and subsequently become negative. Such behavior does *not* arise generically from the geometric action introduced in Sec. V. It would require either a reversal of the large-scale geometric momentum of the embedded spacetime or additional dynamical ingredients beyond the minimal embedding sector considered here. We therefore emphasize that a weakening of cosmic acceleration does *not* imply an impending recollapse. In the embedded-spacetime framework, weakening acceleration corresponds to a slowly evolving geometric attractor that remains strictly non-phantom and expanding.

e. Connection to BAO and supernova likelihood analyses. Observational constraints on late-time expansion are commonly reported in terms of the Hubble rate $H(z)$, the comoving angular-diameter distance $D_M(z)$, and the luminosity distance $D_L(z)$. Baryon acoustic oscillation (BAO) measurements directly constrain

$$D_H(z) \equiv \frac{1}{H(z)}, \qquad D_M(z) = \int_0^z \frac{dz'}{H(z')}, \tag{42}$$

while Type Ia supernova observations probe

$$D_L(z) = (1+z)\,D_M(z). \tag{43}$$

In the stationary-attractor limit,

$$H(z) \simeq H_{\rm stn}, \tag{44}$$

the expansion history is indistinguishable from that of a cosmological constant over the redshift range probed by current BAO and SN data. A weakening-attractor phase corresponds instead to a slowly varying Hubble rate,

$$\frac{d\ln H}{d\ln a} = \mathcal{O}(\epsilon), \qquad |\epsilon| \ll 1, \tag{45}$$

which induces small, smooth deviations from ΛCDM in $D_H(z)$ and $D_M(z)$ without introducing phantom behavior or rapid evolution. In effective-fluid language, one may define

$$w_{\rm eff}(z) \equiv -1 - \frac{2}{3}\frac{d\ln H}{d\ln a}, \tag{46}$$

so that weakening acceleration corresponds to

$$-1 < w_{\rm eff}(z) \lesssim -1 + \mathcal{O}(\epsilon), \tag{47}$$

with a mild redshift dependence. Such behavior lies squarely within the class of models explored in joint BAO and supernova likelihood analyses and does not imply deceleration or future contraction. From the perspective of the embedded-spacetime framework, this observational situation is natural. Geometric acceleration is expected to weaken first through slow drift or small oscillatory modulations of $H(z)$ rather than through abrupt changes of sign. Future BAO measurements with improved precision, therefore, offer a direct probe of the large-scale embedding dynamics without requiring exotic matter components.

V. CONSTRAINT-BASED EMBEDDING DYNAMICS IN CODIMENSION TWO

A central question raised in recent discussions is whether the late-time accelerating attractor described in Sec. V can arise purely from embedding constraints, in the spirit of Regge–Teitelboim (RT) gravity, without invoking explicit rigidity or extrinsic-curvature terms. In this section we analyze this possibility explicitly for the physically relevant case of codimension two, corresponding to two bulk embedding coordinates.

A. RT formulation and embedding constraints

In RT gravity, spacetime is described as a four-dimensional manifold embedded in a flat ambient space, with dynamics governed by the Einstein–Hilbert action expressed in terms of embedding variables,

$$S_{\rm RT} = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-g}\, R[g(X)] + S_{\rm m}. \tag{48}$$

Variation with respect to the embedding functions $X^A(x^\mu)$ yields the RT equations,

$$\nabla_\mu \left(G^{\mu\nu} \partial_\nu X^A \right) = 0, \tag{49}$$

which supplement the usual Einstein equations with additional constraint structure. These equations ensure conservation of the Einstein tensor projected along the embedding directions but introduce no new local propagating degrees of freedom beyond general relativity.

B. Homogeneous codimension-two embeddings

We specialize to homogeneous and isotropic cosmology with codimension two, parameterized by two bulk embedding coordinates,

$$Y(\tau), \qquad Z(\tau), \tag{50}$$

orthogonal to the four-dimensional spacetime manifold. The induced metric on the manifold is of FRW form,

$$ds^2 = -d\tau^2 + a^2(\tau)\, d\Sigma_k^2, \tag{51}$$

with $k = +1$ corresponding to compact spatial sections, as advocated in the constraint-based approach. The RT equations imply conservation of a geometric momentum current associated with translations in the ambient space. For homogeneous embeddings this reduces to two conserved quantities,

$$P_Y = a^3 G^{00} \dot{Y}, \qquad P_Z = a^3 G^{00} \dot{Z}, \tag{52}$$

where overdots denote derivatives with respect to proper time τ. These quantities encode the collective motion of spacetime in the two normal directions.

C. Absence of a late-time accelerating attractor

In the absence of explicit rigidity or extrinsic-curvature terms, the conserved momenta (52) are diluted by the expansion as

$$\dot{Y}, \dot{Z} \propto \frac{1}{a^3 G^{00}}. \tag{53}$$

At late times, when matter and radiation dilute and $G^{00} \sim H^2$ becomes small, this scaling forces the bulk velocities to decay rapidly. As a result, the embedding motion asymptotically freezes rather than sustaining a finite geometric contribution to the expansion. Substituting this behavior into the Friedmann equation yields

$$H^2(\tau) \longrightarrow \frac{1}{3M_{\rm Pl}^2} \left(\rho_{\rm m} + \rho_{\rm r} \right), \tag{54}$$

with no residual constant or slowly varying term. Thus, pure RT constraint dynamics in codimension two do *not* produce a stationary or quasi-stationary accelerating attractor at late times. This result is robust and independent of the choice of spatial curvature k or initial conditions. Without additional geometric structure, the embedding degrees of freedom redshift away too efficiently to support sustained acceleration.

D. Implications for early-time inflation

A similar conclusion applies to the early Universe. While ultrarelativistic bulk motion can generate strong time dilation and mimic inflationary behavior kinematically, constraint-based RT dynamics alone do not stabilize this phase. In codimension two, small deviations from perfect ultrarelativistic motion feed back rapidly through the constraints, preventing a prolonged period of constant physical Hubble rate without additional control parameters. In other words, RT constraints alone permit transient inflation-like behavior

but do not naturally yield a long-lived inflationary phase with controlled exit and predictive perturbations.

E. Role of rigidity or conserved geometric charge

The analysis above clarifies the role played by the extrinsic-curvature sector introduced in Sec. V. Rigidity terms, or an equivalent conserved geometric charge associated with the embedding, prevent the rapid dilution of embedding momentum and allow the system to approach a stationary or slowly drifting attractor,

$$H(\tau) \to H_{\rm stn}(\tau). \tag{55}$$

From this perspective, rigidity should not be viewed as an ad hoc modification but as the minimal ingredient required to stabilize collective embedding dynamics in an expanding universe. We emphasize that the physical requirement is not the specific form of the extrinsic-curvature invariants or the values of the coefficients α and β, but rather the existence of a conserved or weakly evolving geometric quantity capable of supporting late-time acceleration. Any microscopic mechanism—constraint-based or otherwise—that achieves this goal would lead to qualitatively similar cosmological behavior.

F. Summary of the codimension-two test

The codimension-two RT analysis leads to a clear conclusion:

- Constraint-based embedding dynamics alone do not support sustained inflation or late-time acceleration.

- Ultrarelativistic bulk motion can generate transient time-dilated expansion but is not dynamically stabilized.
- A rigidity term or equivalent conserved geometric charge is required to produce the stationary or weakening-attractor behavior compatible with observations.

Thus, while the RT framework provides a powerful geometric foundation, additional large-scale geometric structure is essential for a complete cosmological model. The extrinsic-curvature sector introduced in this work supplies precisely this structure in a minimal and physically transparent manner.

VI. LINEAR PERTURBATIONS FROM EMBEDDING FLUCTUATIONS

In an embedded description of spacetime, perturbations are not restricted to intrinsic metric fluctuations. The embedding itself introduces unavoidable physical degrees of freedom corresponding to deformations of the spacetime manifold in directions normal to the embedding. These co-dimensional modes are absent in purely intrinsic formulations of gravity and play a central role in the present framework. In this section we show that embedding fluctuations provide a natural origin for primordial cosmological perturbations during inflation and lead to distinctive predictions for scalar and tensor modes.

A. Normal fluctuations of the embedding

We expand the embedding functions around a homogeneous background configuration $\bar{X}^A(x^\mu)$ as

$$X^A(x^\mu) = \bar{X}^A(x^\mu) + n_I^A(x^\mu)\,\xi^I(x^\mu) + \cdots\,, \qquad (56)$$

where n_I^A are orthonormal unit vectors normal to the background spacetime manifold and $\xi^I(x^\mu)$ are scalar fields describing transverse displacements of the manifold in the ambient space. These fields are invariant under reparametrizations of the intrinsic coordinates and therefore represent genuine physical degrees of freedom. At linear order, the ξ^I encode bending of spacetime in the ambient space and are distinct from the usual scalar, vector, and tensor perturbations of the induced metric.

B. Quadratic action for embedding modes

Expanding the geometric action to quadratic order in the embedding fluctuations yields an effective action of the form

$$S^{(2)} = \frac{1}{2}\sum_I \int d\tau\, d^3x\, a^3(\tau)\left[\dot{\xi}_I^2 - \frac{(\nabla\xi_I)^2}{a^2} - m_{\rm eff}^2\,\xi_I^2\right], \qquad (57)$$

where overdots denote derivatives with respect to proper time τ and $m_{\rm eff}^2$ is an effective mass determined by the background extrinsic curvature and geometric rigidity parameters. Generically, $m_{\rm eff}^2 = \mathcal{O}(H^2)$. Fourier decomposing,

$$\xi_I(\tau,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\,\xi_{I,k}(\tau)\,e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad (58)$$

each mode satisfies

$$\ddot{\xi}_{I,k} + 3H\,\dot{\xi}_{I,k} + \left(\frac{k^2}{a^2} + m_{\rm eff}^2\right)\xi_{I,k} = 0. \qquad (59)$$

C. Behavior during the inflationary phase

During the inflationary phase driven by embedding-induced time dilation, the background evolution is approximately de Sitter,

$$H(\tau) \simeq H_{\rm inf} = \text{const.} \qquad (60)$$

Provided $m_{\rm eff}^2 \ll H_{\rm inf}^2$, the embedding fluctuations behave as light scalar fields. Modes with physical wavelength $k/a \gg H_{\rm inf}$ oscillate as in flat space, while modes with $k/a \ll H_{\rm inf}$ freeze out with nearly constant amplitude. The resulting power spectrum for each light embedding mode is

$$\mathcal{P}_\xi(k) \simeq \left(\frac{H_{\rm inf}}{2\pi}\right)^2\left(\frac{k}{k_*}\right)^{n_\xi - 1}, \qquad (61)$$

with spectral tilt

$$n_\xi - 1 \simeq -2\epsilon_H - \frac{2}{3}\frac{m_{\rm eff}^2}{H_{\rm inf}^2}, \qquad (62)$$

where $\epsilon_H \equiv -\dot{H}/H^2$.

D. Conversion to curvature perturbations

Embedding fluctuations modulate the local expansion history by inducing small variations in the lapse and embedding-time expansion rate. This leads to curvature perturbations through a geometric analogue of the δN mechanism. At leading order, the comoving curvature perturbation ζ is

$$\zeta \simeq \sum_I \frac{\partial N_e}{\partial \xi_I}\,\delta\xi_I, \qquad (63)$$

where N_e is the number of inflationary e-folds. The resulting curvature power spectrum is

$$\mathcal{P}_\zeta(k) \simeq \left(\frac{H_{\rm inf}}{2\pi}\right)^2\left(\frac{k}{k_*}\right)^{n_s - 1}, \qquad (64)$$

with

$$n_s - 1 \simeq -2\epsilon_H - \frac{2}{3}\gamma, \qquad \gamma \equiv \frac{m_{\rm eff}^2}{H_{\rm inf}^2}. \qquad (65)$$

E. Tensor perturbations

Tensor perturbations correspond to intrinsic transverse–traceless fluctuations of the induced metric. Because inflation in this framework is driven primarily by time-dilation effects rather than by a large intrinsic energy density, tensor modes are generically suppressed. Parametrically, the tensor-to-scalar ratio takes the form

$$r \sim \frac{H_{\rm inf}^2}{M_{\rm Pl}^2}\,\mathcal{S}, \tag{66}$$

where $\mathcal{S} \ll 1$ is a suppression factor reflecting the weak coupling between bulk-induced kinematics and intrinsic tensor modes. The tensor-to-scalar ratio is naturally low, $r \ll 0.01$.

F. Late-time embedding modes

Ultra-long-wavelength embedding fluctuations can persist to late times and reappear as co-dimensional modes modulating the expansion history. Such modes contribute a subdominant oscillatory component to the Hubble rate,

$$H(\tau) = H_{\rm stn} + \varepsilon\cos(\omega\tau + \phi), \tag{67}$$

with $\varepsilon \ll H_{\rm stn}$. Observation of such oscillatory signatures would provide a direct probe of embedding dynamics, potentially producing detectable coherent ripples in low-redshift distance-redshift relations distinguishable from ΛCDM by joint analyses of current and upcoming data (e.g., DESI, Euclid).

VII. RELATION TO OTHER BRANE-WORLD AND EMBEDDING SCENARIOS

The framework developed in this work involves extra dimensions and an embedded description of spacetime, and it is therefore important to clarify how it differs from other higher-dimensional and brane-world approaches to cosmology. Although these frameworks share certain geometric ingredients, their physical interpretation and cosmological consequences are fundamentally distinct.

A. Comparison with ADD and Randall–Sundrum models

In conventional brane-world scenarios such as the Arkani-Hamed–Dimopoulos–Dvali (ADD) model and the Randall–Sundrum (RS) constructions, extra dimensions are introduced primarily to modify the gravitational sector. In ADD models, large flat extra dimensions lower the effective Planck scale and address the hierarchy problem, while cosmological evolution is governed by modified Friedmann equations arising from higher-dimensional gravity. In Randall–Sundrum models, warped extra dimensions and brane tension localize gravity and generate an effective four-dimensional description.

In all of these scenarios, cosmic acceleration—when present—is sourced by vacuum energy, brane tension, bulk curvature, or explicit matter fields. Extra dimensions alter the *strength* or localization of gravity rather than introducing new kinematic degrees of freedom for spacetime itself.

By contrast, the framework presented here assumes a flat ambient Minkowski bulk and does not rely on brane tension, warped geometries, or bulk curvature. The gravitational coupling on the spacetime manifold is not modified by the presence of extra dimensions. Instead, the essential new physics arises from the fact that spacetime itself is treated as a dynamical object capable of collective motion and deformation in the normal directions of the ambient space. As a result, cosmic acceleration in the present framework is not sourced by vacuum energy or brane tension. Inflation arises from embedding-induced time dilation associated with ultrarelativistic bulk motion, while late-time acceleration emerges from large-scale embedding dynamics and geometric attractors. Extra dimensions therefore modify cosmic *kinematics* rather than the gravitational field equations.

B. Relation to Regge–Teitelboim embedding gravity

The present work is more closely related in spirit to Regge–Teitelboim (RT) embedding gravity, in which general relativity is formulated as a theory of a four-dimensional spacetime surface embedded in a higher-dimensional flat space. RT gravity demonstrates that Einstein's equations can arise from an embedding principle subject to additional constraints, without introducing new local degrees of freedom at the level of intrinsic geometry.

However, the cosmological implications of the embedding degrees of freedom are largely unexplored in the RT framework. In particular, the distinction between embedding time and proper time, and the possibility of strong time-dilation effects, play no explicit role in standard RT treatments. Similarly, long-wavelength collective modes of the embedding are typically constrained away or treated as gauge artifacts.

In contrast, the framework developed here treats the embedding degrees of freedom as physically relevant and dynamically active. Two features are especially important: (i) the explicit separation between embedding time and proper time, which allows strong time-dilation effects and leads naturally to inflation, and (ii) the existence of ultra-long-wavelength co-dimensional modes that can persist over cosmological timescales and influence late-time expansion.

While rigidity terms or equivalent geometric structures are natural in effective descriptions of embedded manifolds, our key results do not depend sensitively on the detailed microscopic origin of these terms. Whether the late-time geometric attractor arises from explicit extrinsic-curvature invariants or from conserved quantities associated with embedding constraints, in the spirit of RT gravity, remains an open question.

C. Conceptual distinction and physical interpretation

The essential distinction between the present framework and conventional brane-world cosmologies can be summarized succinctly. In standard brane-world models, extra dimensions modify the gravitational sector by altering coupling strengths, introducing new energy scales, or changing bulk geometry. In the embedded-spacetime framework developed here, extra dimensions modify the allowed *motions* of spacetime itself. Cosmic acceleration then appears not as an additional energy component but as a geometric consequence of embedding kinematics and dynamics.

Inflation, late-time acceleration, and possible oscillatory signatures correspond to different dynamical regimes of a single underlying structure: a four-dimensional spacetime manifold undergoing collective motion and deformation in a higher-dimensional flat background. This perspective allows phenomena that are traditionally treated as unrelated—inflation, dark energy, and long-wavelength anomalies—to be understood within a unified geometric framework.

VIII. DISCUSSION AND CONCLUSIONS

We have developed a cosmological framework in which both early- and late-time cosmic acceleration arise from the geometry and kinematics of spacetime embedded in a higher-dimensional flat background. In this picture, spacetime is not merely a four-dimensional manifold endowed with intrinsic curvature, but a dynamical surface capable of collective motion and deformation in co-dimensional directions. Intrinsic and extrinsic geometry are inseparably linked, and the embedding degrees of freedom play a direct physical role in cosmic evolution.

A central result of this work is the demonstration that a genuine phase of inflation can arise purely from embedding-induced time dilation. When the spacetime manifold undergoes ultrarelativistic motion in transverse directions of the ambient space, proper time along the manifold advances very slowly relative to embedding time. Provided the embedding-time expansion rate remains approximately constant, this kinematic effect leads to *strict exponential growth* of the scale factor with respect to proper time,

$$a(\tau) \propto e^{H_{\rm inf}\tau}, \tag{68}$$

as derived explicitly in Sec. III. This inflationary phase satisfies the defining criterion of inflation, including a shrinking comoving Hubble radius, and resolves the horizon and flatness problems without invoking an inflaton field, scalar potential, or slow-roll dynamics. A graceful exit arises naturally as bulk velocities decrease and proper time resumes normal flow.

Following inflation, the Universe transitions smoothly to a standard radiation- and matter-dominated Friedmann–Robertson–Walker phase. Although the lapse relaxes to order unity, the embedding geometry retains memory of its early evolution. This memory is encoded in large-scale geometric quantities and weakly excited co-dimensional modes seeded during the exit from inflation.

At late times, once conventional energy densities have sufficiently diluted, the large-scale embedding dynamics become dynamically relevant. We have shown that a geometric expansion attractor generically emerges, characterized by a stationary or quasi-stationary Hubble rate,

$$H(\tau) = H_{\rm stn} + H_{\rm osc}(\tau), \tag{69}$$

where $H_{\rm stn}$ sets the mean expansion timescale and $H_{\rm osc}$ represents a subdominant oscillatory modulation. The existence of this attractor provides sustained late-time acceleration without invoking a bare cosmological constant or vacuum energy. Importantly, a slow weakening of the acceleration does not imply an eventual transition to contraction; contraction requires the stationary component itself to cross zero, which does not occur generically in the embedding dynamics studied here.

The framework admits a natural theory of cosmological perturbations. Transverse embedding fluctuations behave as light scalar degrees of freedom during inflation and generate nearly scale-invariant curvature perturbations via a geometric analogue of the δN mechanism. The scalar spectral tilt is controlled by slow variation of the Hubble rate and by dimensionless geometric parameters, while tensor perturbations are generically suppressed. This suppression follows from the fact that inflation is driven by bulk kinematics rather than large intrinsic energy density, and from the weak coupling between ultrarelativistic embedding motion and intrinsic transverse–traceless modes.

An important conceptual distinction between the present framework and conventional brane-world or modified-gravity scenarios is that the ambient space is flat and extra dimensions do not modify gravitational couplings or introduce new energy scales. Instead, extra dimensions enlarge the space of allowed *motions* of spacetime itself. Inflation, late-time acceleration, and possible oscillatory signatures then correspond to different dynamical regimes of a single underlying structure: a four-dimensional spacetime manifold undergoing collective motion and deformation in a higher-dimensional flat background.

Several open questions remain. First, while we introduced a minimal effective action containing extrinsic curvature terms to capture late-time acceleration, the microscopic origin of this geometric sector is not yet specified. In particular, it remains to be determined whether the late-time attractor can arise entirely from constraint dynamics or conserved geometric quantities, in the spirit of Regge–Teitelboim embedding gravity, without invoking explicit rigidity terms.

Second, a detailed treatment of reheating and matter production following the inflationary phase is required to complete the cosmological history. Third, more precise predictions for non-Gaussianities, mode coupling, and late-time observables will be necessary to fully confront the framework with data.

Despite these open issues, the results presented here establish that embedding geometry alone is sufficient to reproduce the key qualitative features of the observed Universe. Inflation, primordial structure, and late-time acceleration need not be attributed to separate exotic components but can instead emerge from the kinematics and dynamics of spacetime itself. This perspective suggests that the embedding of spacetime may play a more fundamental role in cosmology than previously appreciated and motivates further investigation of embedded-manifold dynamics as a foundation for cosmic evolution.

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